Effect of mathematical modeling on the estimation of critical power

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ABSTRACT

BULL, A. J., T. J. HOUSH, G. O. JOHNSON, and S. R. PERRY. Effect of mathematical modeling on the estimation of critical power. Med. Sci. Sports Exerc., Vol. 32, No. 2, pp. 526 –530, 2000. Purpose: The purposes of this study were to re-examine the findings of previous studies by comparing the critical power (CP) estimates from five mathematical models and to determine the time to exhaustion during cycle ergometry at the lowest CP estimate from the five models. Methods: Nine adult males performed a maximal incremental test to determine peak power and five or six randomly ordered trials on a cycle ergometer for the estimation of CP. Two linear, two nonlinear, and one exponential mathematical model were used to estimate CP. The subjects then completed two trials to exhaustion, or 60 min, at their lowest estimate of CP from the five models. Results: The nonlinear three-parameter model (Nonlinear-3) produced a mean CP that was significantly (P < 0.05) less than the mean CP values derived from the other four models and was the lowest CP estimate for each subject. Two and three subjects, however, did not complete 60 min of cycling during the first and second trials at CP, respectively. At the end of the trials the subjects who completed 60 min of cycling had a mean heart rate of 92% of their maximum and a mean rating of perceived exertion of 17. Conclusion: These findings support previous studies that have indicated that in many cases CP overestimates the power output that can be maintained for at least 60 min. Key Words: CYCLE ERGOMETRY, TIME TO EXHAUSTION, HEART RATE, RPE

Monod and Scherrer (18) developed a technique for determining “the amount of work a muscle can do before being exhausted” and “the conditions of a fatigueless task” which is called the “critical power (CP) test.” Theoretically, the CP test provides estimates of the maximal rate of fatigueless work (CP) and the total amount of work associated with the “energetic reserves” stored in the muscle (anaerobic work capacity (AWC)) (4,18). Moritani et al. (19) proposed a whole-body analog of the CP test using a series of exhaustive work bouts at various power outputs (P) on a cycle ergometer. Like Monod and Scherrer (18), Moritani et al. (19) found highly linear (r² = 0.982–0.998) relationships between total work (TW) and time to exhaustion (t) for the cycle ergometer work bouts.

More recent investigations (13,15–17) have found that the CP test overestimates the P that can be maintained for “a long period of time” (18). In general, these studies have reported that many subjects were able to cycle for less than 60 min at their CP. Therefore, to identify more valid estimates of CP, various mathematical expressions of the P versus t relationship have been evaluated (8,9,11,25).

Recently Gaesser et al. (8) examined CP estimates from five mathematical models for describing the P versus t relationship during cycle ergometry. In addition to the TW versus t model (Linear-TW) proposed by Moritani et al. (19), Gaesser et al. (8) estimated CP from a second linear model (Linear-P), a two-parameter nonlinear model (Nonlinear-2), a three-parameter nonlinear model (Nonlinear-3), and an exponential model (EXP). It was reported (8) that the Nonlinear-3 model of Morton (20) produced CP estimates that were significantly (P < 0.05) less than the other four models and not significantly (P > 0.05) different from the ventilatory threshold for long-term exercise (LTE TVENT). No studies, however, have determined t for cycling at the lowest CP estimate from the various models.

Therefore, the purpose of this study was two-fold: 1) to re-examine the findings of Gaesser et al. (8) by comparing the CP estimates from the five mathematical models, and 2) to determine t during cycle ergometry at the lowest CP estimate from the five models.

METHODS

Subjects. Nine male subjects (mean ± SD age = 25 ± 3 yr, height = 177 ± 6 cm, weight = 81.6 ± 9.4 kg) who...
were not highly experienced cyclists volunteered for this study. All procedures were approved by the University Institutional Review Board for Human Subjects and all subjects signed a written informed consent form before testing.

**Cycle ergometer trials.** Each subject completed eight or nine trials, with $\geq 24$ h separating each trial. The subjects first performed a maximal incremental test to exhaustion on an electronically braked cycle ergometer (Corval 400, Quinton Instruments, Seattle, WA) maintaining 60 rpm as closely as possible. The maximal test began with a warm-up period of 2 min with no resistance. Thereafter, the power output was increased 30 W each min until the subject could no longer maintain a pedal cadence of 60 rpm despite verbal encouragement. The power output and heart rate attained at exhaustion were considered to be the subject’s peak power ($P_{\text{peak}}$) and peak heart rate ($HR_{\text{peak}}$) (Polar Pacer, Polar CIC, Inc., Port Washington, NY).

The subjects then performed five or six randomly ordered trials at 60 rpm at power outputs ranging from $P_{\text{peak}}-130$ W to $P_{\text{peak}}+50$ W for the estimation of CP. The first five predictive trials were between $P_{\text{peak}}-50$ W and $P_{\text{peak}}+50$ W. If a subject did not have a trial lasting approximately 10 min, the subject was given a sixth trial at a power output estimated to allow more than 10 min of cycling based on their previous trials. The warm-up for each trial included 4 min of cycling at 30 W. The power output was then increased to the predetermined level and the timing started. The trials were ended and $t$ was recorded to the nearest second when the subject could no longer maintain 60 rpm despite verbal encouragement. The subjects were allowed to view the cadence meter but were not informed of the power output or elapsed time of any trial.

**Mathematical modeling.** Five regression models (Fig. 1) from Gaesser et al. (8) were used to estimate CP.

The Linear-TW model, $TW = AWC + CP \cdot t$, was based on the regression of TW versus $t$, utilizing the equation of a line: $y = a + bx$, where $TW = \text{total work performed}$, $AWC = \text{anaerobic work capacity}$, $CP = \text{critical power}$, and $t = \text{time to exhaustion}$ (7,9,13,16,18,19,21,25).

A second, mathematically equivalent, linear model (Linear-P) was derived by solving for $P$, utilizing two equations for total work: $TW = P \cdot t$ and $TW = AWC + CP \cdot t$.

$$P \cdot t = AWC + CP \cdot t$$

divided by $t$ yields $P = AWC/(1/t) + CP$.

where $P = \text{power}$; and $1/t$ = the inverse of $t$. By using the inverse of $t$ $(1/t)$, one can convert the hyperbolic relationship between $P$ and $t$ into linear form (8,9,13,17,18,23,25).

A third mathematically equivalent but nonlinear model was also used to estimate CP. This model was based on the hyperbolic relationship between $P$ and $t$ (8,9,18,24,25) and can be described by the Nonlinear-2 equation: $t = AWC/(P - CP)$, which can be derived by solving the Linear-P equation for $t$. Thus, the derivation is:

$$P = AWC/(1/t) + CP,$$

minus CP yields, $P - CP = AWC/t$, 

The final Nonlinear-3 model becomes

$$t = (AWC/(P - CP)) - (AWC/(P_{\text{max}} - CP)).$$

The fifth regression model was an exponential model (EXP): $P = CP + (P_{\text{max}} - CP) \exp(-t/\tau)$; where $\tau$ = an undefined time constant (8,12). Like the Nonlinear-3 model, this model provides a $P_{\text{max}}$ value (an x-intercept of the power axis) to overcome the assumption of infinite power at very short

![Figure 1—Regression analyses and curve fits for each of the five mathematical models used to estimate CP for one representative subject. The models are described in the text.](Image 527)

multiplied by $t$ yields $(P - CP) \cdot t = AWC$,

divided by $(P - CP)$ yields $t = AWC/(P - CP)$.

The fourth model is also a nonlinear model but includes the parameter maximal instantaneous power ($P_{\text{max}}$) (8,20). The $P_{\text{max}}$ parameter was added to overcome the assumption of the Nonlinear-2 model that as time approaches zero, power is infinite (8,20). $P_{\text{max}}$ allows for a time asymptote that is below the x axis (power axis) where time = 0 and provides a $P_{\text{max}}$ value at the x-intercept (20). The mathematical addition of $P_{\text{max}}$ and the derivation of the Nonlinear-3 model is made possible by the addition of a third parameter, $k$, to the Nonlinear-2 model (20). The parameter $k$ allows a nonzero time asymptote at $t = k$ by writing the Nonlinear-2 model in the form:

$$t = AWC/(P - CP) + k;$$

and then developing a relationship between $k$ and $P_{\text{max}}$, by allowing

$$P = P_{\text{max}} \text{ at } t = 0, \text{ so that:}$$

$$k = -(AWC/(P_{\text{max}} - CP))$$

The fifth regression model becomes

$$t = (AWC/(P - CP)) - (AWC/(P_{\text{max}} - CP)).$$

The fifth regression model was an exponential model (EXP): $P = CP + (P_{\text{max}} - CP) \exp(-t/\tau)$; where $\tau$ = an undefined time constant (8,12). Like the Nonlinear-3 model, this model provides a $P_{\text{max}}$ value (an x-intercept of the power axis) to overcome the assumption of infinite power at very short
durations. However, the equation does not provide an estimation of AWC (8,12).

**Trials at critical power.** The mathematical model yielding the lowest CP estimate for each subject was used to determine the P for the two trials at CP (CP-1 and CP-2). The trials began with a warm-up of 4 min of cycling at 30 W and 60 rpm. The P was then increased to the estimated CP and the timing was started. The trials were terminated and t was recorded to the nearest second when pedal cadence could not be maintained at 60 rpm, or the subject completed 60 min. The subjects’ ratings of perceived exertion (RPE) (2) and heart rates (HR) were recorded every 3 min during CP-1 and CP-2.

**Statistical analysis.** For each subject the P and t data were fit to the five mathematical models to estimate CP. The coefficients of determination (r²) and the SE of the CP estimates were calculated to examine the goodness of fit of the data to the models (SPSS, SPSS Inc., Chicago, IL). The SE values reported in this investigation are the standard deviations of the sampling distributions of the CP estimates from the five models (22). Therefore, for the Linear-TW model the SE reported in this study is the SE of the slope coefficient. For the Linear-P model, however, it is the SE of the y-intercept. For the Nonlinear-2, Nonlinear-3, and EXP models the SE values are for the asymptotes of the P versus t relationships. A repeated measures analysis of variance (ANOVA) (1 × 5) with Scheffe post-hoc comparisons were used to determine differences in mean CP between the five models and paired t-tests were used to determine differences in mean HR values at 45 versus 60 min for CP-1 and CP-2 (Statview 512+, Brain Power, Inc., Calabasas, CA). All values are presented as mean (± SD) and an alpha of P < 0.05 was considered statistically significant for all analyses.

**RESULTS**

**Critical power estimates.** The r² and SE values for the five mathematical models used to estimate CP ranged from: Linear-TW, r² = 0.997–1.000, SE = 1–5 W; Linear-P, r² = 0.847–0.991, SE = 6–15 W; Nonlinear-2, r² = 0.894–1.000, SE = 0–8 W; Nonlinear-3, r² = 0.937–1.000, SE = 0–39 W; EXP, r² = 0.904–0.996, SE = 5–20 W. The AWC values ranged from: Linear-TW, 9.6–26.5 kJ; Linear-P, 5.4–20.6 kJ; Nonlinear-2, 11.1–34.9 kJ; and Nonlinear-3, 16.6–68.0 kJ. The Pmax values ranged from 357–803 W for the Nonlinear-3 model and 337–562 W for the EXP model. The duration of the shortest trials used to estimate CP ranged from 0.6 to 2.2 min and the longest trials ranged from 9.0 to 35.8 min. The mean CP estimates of the five mathematical models are presented in Table 1. The repeated measures ANOVA indicated that there was a significant difference among the mean CP estimates for the five models, and the results of the Scheffe post-hoc comparisons (Table 1) indicated that the Nonlinear-3 model produced a significantly lower mean CP estimate than the other models. Furthermore, the Nonlinear-3 model resulted in the lowest estimate of CP for each subject.

**Time to CP trial termination.** Of the nine subjects, two did not complete 60 min of cycling during CP-1 or CP-2. The t values for these two subjects were 45.0 and 20.9 min for CP-1 and 48.1 and 18.0 min for CP-2. One subject cycled for 60 min during CP-1 but completed only 53.6 min during CP-2. The five remaining subjects completed 60 min during both trials at CP.

**HR at CP trial termination.** Table 2 shows the mean HR responses of the subjects who completed 60 min during the trials at CP. The seven subjects who completed 60 min during CP-1 had a mean HR of 166 ± 10 beats·min⁻¹ at completion, which represented 93 ± 5% of HRpeak. At the completion of CP-2, the six subjects had a mean HR of 165 ± 12 beats·min⁻¹, which represented 91 ± 5% of HRpeak. Paired t-tests indicated that the increase in mean HR between 45 and 60 min was significant for both CP-1 and CP-2 for these subjects. The mean HR at exhaustion for the three subjects who did not complete 60 min during CP-2 was 175 ± 7 beats·min⁻¹ (97 ± 6% HRpeak).

**RPE at CP trial termination.** Table 2 shows the mean RPE values of the subjects completing 60 min during the trials at CP. The mean RPE values for those subjects completing 60 min during CP-1 and CP-2 were 19 ± 1 and 17 ± 3, respectively. The mean RPE value at exhaustion for the two subjects who did not complete 60 min during CP-1 was 175 ± 7 beats·min⁻¹ (97 ± 6% HRpeak).

**DISCUSSION**

In the present investigation, the individual r² values for the various mathematical models ranged from 0.847 to 1.00. These findings were consistent with previous studies that have reported values from r² = 0.84 to 1.00 and demonstrated a close relationship between P and t irrespective of the mathematical modeling (4,8,11,13,15–17,19,23,25). Furthermore, like the study by Gaesser et al. (8), the present study showed that the Linear-P model tended to result in lower r² values than the other four models used to estimate CP. It has been suggested that this may result from the conversion of the hyperbolic P versus t relationship to the linear TW versus t and P versus the inverse of t (1/t) relationships (10,20,26). Gaesser et al. (8, p. 1431) have
stated that “... statistical analysis of nonlinear data after transformation of the data to linear form often produces parameter estimates different from the results of nonlinear analysis of the data” and that (p. 1437) “... nonlinear data should be treated with nonlinear analysis. ...” The present findings also support the suggestion of Gaesser et al. (8) that the selection of the independent and dependent variables, the expression of the variables (i.e., \( t \) rather than \( 1/t \)), and the number of variables used in the various models can affect the resulting \( r^2 \) values and CP estimates.

The results of the present investigation indicated that there were significant mean differences among the models used to estimate CP. Of the five models examined, the Nonlinear-3 model of Morton (20) resulted in the lowest mean estimate of CP (Table 1) and also the lowest CP estimate for each subject. These findings were consistent with those of Gaesser et al. (8) who also found that the Nonlinear-3 model resulted in estimates of CP that were lower than the estimates obtained using the other four models. Furthermore, Morton (20) found that the Nonlinear-3 model resulted in significantly \( (P < 0.05) \) lower CP estimates than the Nonlinear-2 model and suggested that the Nonlinear-3 model overcomes physiological assumptions inherent in the Nonlinear-2 model including: 1) as \( t \) approached zero, power is infinite, and 2) at exhaustion, all of the muscular energy reserves associated with AWC are exhausted.

The validity of CP derived from the various models as a measure of the maximal rate of fatiguability has been examined by comparing it with other fatigue threshold parameters (7,8,14,16,17,19,23). For example, CP has been validated against the ventilatory anaerobic threshold (T\(_{VENT}\)) (7,19,23), lactate accumulation (OBLA) (14,16), and the ventilatory threshold (\( T_{VENT} \)) (8), and individual anaerobic threshold (IAT) (17).

deVries et al. (7), Gaesser et al. (8), and Moritani et al. (19) found that the power output or oxygen consumption rate (V\(_{O2}\)) at CP (from the Linear-TW (7,19) and Nonlinear-3 models (8)) was not significantly \( (P > 0.05) \) different from that of T\(_{VENT}\) or LTE T\(_{VENT}\). Poole et al. (23), however, found that the power output at CP (Linear-P model) was significantly \( (P < 0.05) \) greater than T\(_{VENT}\). deVries et al. (7) also compared CP (Linear-TW model) with the EMG\(_{FT}\) and found the power output at CP to be significantly \( (P < 0.05) \) less than that at EMG\(_{FT}\).

TABLE 2. Mean (± SD) HR and RPE values for those subjects completing 60 min of cycling during CP-1 (N = 7) and CP-2 (N = 6).

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Poole et al. (23) reported that all eight of their subjects completed 24 min of cycling at CP (Linear-P model) with no...
significant ($P > 0.05$) increase in VO$_2$ during the final 4 min. The subjects, however, had blood lactate levels (5.6 ± 0.9 mmol $z$) that were greater than OBLA. McLellan and Cheung (17) also estimated CP from the Linear-P model and found that only one of their 14 subjects could complete 30 min of cycling (mean ± SD = 20.5 ± 4.5 min). A number of studies (13,15,16,21) have found that the Linear-TW model also overestimates the maximal rate of fatigueless work during cycle ergometry. For example, Housh et al. (13) found that the mean time to exhaustion at CP was 33.3 ± 15.37 min. Jenkins and Quigley (15,16) found that their subjects could not cycle at CP for 30–40 min and that to maintain the trials without exhaustion the power outputs had to be decreased by 4.7 to 5.7%. Overend et al. (21) found that their subjects could complete 24 min of cycling at CP (Linear-TW model), although, like the subjects of Poole et al. (23), blood lactate levels were substantially above OBLA (8.2 and 6.5 mmol $z$ in young and elderly groups, respectively). Scarborough et al. (24) used the Nonlinear-2 model and reported that the subjects could cycle at CP for a mean of 42.9 ± 6.6 min in a first trial and 50.8 ± 6.9 min in a second trial. While these studies (13,15–17,24) indicate that CP from the various mathematical models overestimate the maximal rate of fatigueless work, Bishop et al. (1) have recently shown that the durations of the predictive trials affect the estimation of CP, and Hill et al. (11) have shown that the CP estimate is also affected by the choice of pedal cadence. Future studies should examine further the effects of the duration of the predictive trials and pedal cadence on CP estimates and time to exhaustion at CP from the five mathematical models used in the present study.

The current study found that 22–33% of our subjects could not complete 60 min at CP estimated by the Nonlinear-3 model. This was true even though the Nonlinear-3 model provided the lowest CP estimate for each individual subject and a mean CP estimate that was significantly lower than the mean CP values derived from the other models (Table 1).

Blood lactate, ventilatory, and time to exhaustion data from previous investigations (13–17,21,23,24) suggest that in many cases CP derived from the Linear-TW, Linear-P, and Nonlinear-2 models overestimates the power output that can be maintained for at least 60 min without exhaustion. The present study supports previous investigations (13–17,21,23,24) that have indicated that CP does not represent a “fatigueless task” as proposed by Monod and Scherrier (18).

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